Simplex Algorithm

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Many STEM majors would be well-suited to study optimization or, more broadly, industrial engineering, in graduate school, but most are not aware of this exciting, in-demand field. We envision Linear Algebra students as the primary target for this module because students who are studying Linear Algebra are also a likely audience for optimization, and because Linear Algebra is so fundamental to the study of optimization. We seek to introduce the field of optimization to these students through a series of engaging activities that both reinforce concepts from Linear Algebra and develop a conceptual framework that will be useful for students who go on to study optimization in a later course.

The mathematical concept of optimization is one that is used in everyday life constantly. Many everyday situations (financial, economic, business, etc.) can be modeled as optimization problems. The mathematical field of optimization offers various principles and methods for solving quantitative problems that require maximizing or minimizing a quantity. This module aims to introduce some of these methods and algorithms (such as the linear optimization and the simplex algorithm) to an undergraduate audience. Specific class activities are suggested to be performed that will help students become familiar with the optimization in real-life problems.

The first activity:
Each student will require 12 large Lego blocks (rectangular top with 8 connectors) and 18 small Lego blocks (square top with 4 connectors).
Problem 1. Imagine that you are running a manufacturing business that makes tables and chairs out of Legos. The tables can be sold for \$16 each and the chairs are sold for \$10 each. Each table requires 2 big blocks and 2 little blocks while each chair requires 1 big block and 2 little blocks, as shown below. There are 12 big and 18 small blocks available to build tables and chairs.

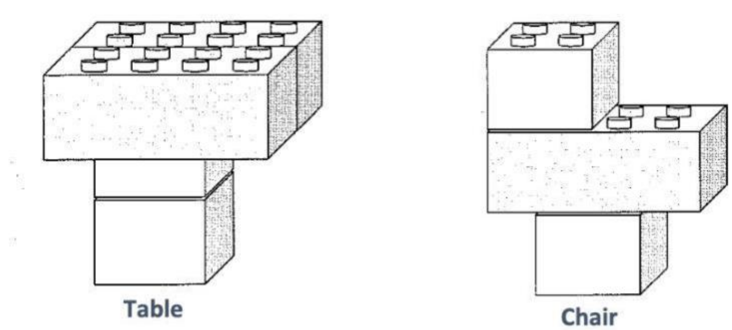


Figure 1 – Lego table and chair

What is the maximum revenue that you can achieve with these resources? In other words, how many tables and chairs would you build, and how much revenue would this generate?

Solution: Students will likely begin by building 6 tables for a revenue of $96, because the revenue from a table is higher than the revenue from a chair. They will have some small blocks leftover and realize that they can take apart one table to build two chairs, increasing their revenue by $4. Doing this twice more leads to the optimal solution: 3 tables and 6 chairs for a revenue of $108.

The Legos problem can be modeled using a linear optimization model, or linear program, a name that is often shortened to "LP". Linear programs have three main components: decision variables, an objective function, and constraints.

The decision variables in a linear program are variables that represent the decisions to be made. In this problem, we seek to determine how many tables and chairs to make. Thus, we define decision variables

T := the number of tables to make;

C := the number of chairs to make.

Now we are ready to define our objective function: a function of the decision variables that represents the value that we wish to maximize or minimize. In this case, we wish to maximize revenue, which can be expressed as,

Maximize Revenue := $16T + 10C$

Finally, we must define the constraints, which are equations or inequalities that impose real world require-ments on the values that the decision variables can take. In the Legos problem, we are restricted to using no more than 12 big and 18 small blocks. Each table requires 2 big blocks and each chair requires 1 big block, so we can express the number of big blocks that we use to make our furniture as $2T+C$. Thus, we can express the limit on the number of available big blocks as, $2T+C\leq 12$. Similarly, we can express the limit on the number of small blocks that are available for our furniture making as $2T+2C\leq 18$.

The constraints limiting the number of available blocks are general constraints. Most linear programs also have variable bounds, which are a class of constraints that limit the range of the decision variables to a continuous interval of real numbers. In this case, it does not make sense to make a negative number of tables or chairs, so we add the so-called nonnegativity constraints, $T\geq 0$, $C\geq 0$.

Now we can write the linear program (LP) for the Legos problem:

You may be thinking, "Don't we also need to make whole numbers of tables and chairs?" This is true, of course. However, in addition to the objective function and constraints being linear, another requirement of a linear programming model is that the variables are allowed to take on any value in a continuous interval of real numbers (subject to the general constraints). A linear optimization model in which the variables are required to take on integer values is called an integer program (IP). Integer programs require a different, much more computationally expensive, algorithm. Thus, we allow fractional numbers of tables and chairs in our solutions.

The Feasible Region

A *solution* to a linear program is an assignment of a number to each of the decision variables. For example, (T,C)=(6,0) is a solution to the Legos problem that represents making six tables and no chairs.

A *feasible solution* to a linear program is a solution that satisfies all of the constraints. The solution (T,C)=(6,0) to the Legos problem is a feasible solution because it satisfies all four constraints: it does not overuse large or small blocks (constraints 1 and 2), and it does not require making a negative number of tables or chairs (constraints 3 and 4).

An *infeasible solution* is a solution that is not feasible; that is, a solution is infeasible if it violates at least one constraint.

The *value* of a solution is the value of the objective function at that solution.

A constraint is *active* at a particular solution if the constraint is satisfied with equality. Constraints 1 and 4 are active for solution (6,0).

Problem ##. For each of the solutions, (T,C), to the Legos problem below: (i) Determine whether the solution is feasible or infeasible. (ii) If the solution is feasible, determine which constraints in the Lego LP are active. If the solution infeasible, determine which constraints are violated. (iii) Find the value of the solution.

A. (4,4)
B. (2,8)
C. (-1,10)
D. (0,9)

A. (i) Feasible, (ii) $2T+C\leq12$ is active, (iii) 104
B. (i) Infeasible, (ii) $2T+2C\leq18$ has been violated, (iii) 112
C. (i) Infeasible, (ii) $T\geq0$ has been violated, (iii) 84
D. (i) Feasible, (ii) $2T+2C\leq18$ and $T\geq0$ are active, (iii)

For two-dimensional linear programs (those with two variables), we can visualize the set of points that satisfy each constraint in the Cartesian plane. A 2-dimensional linear inequality, such

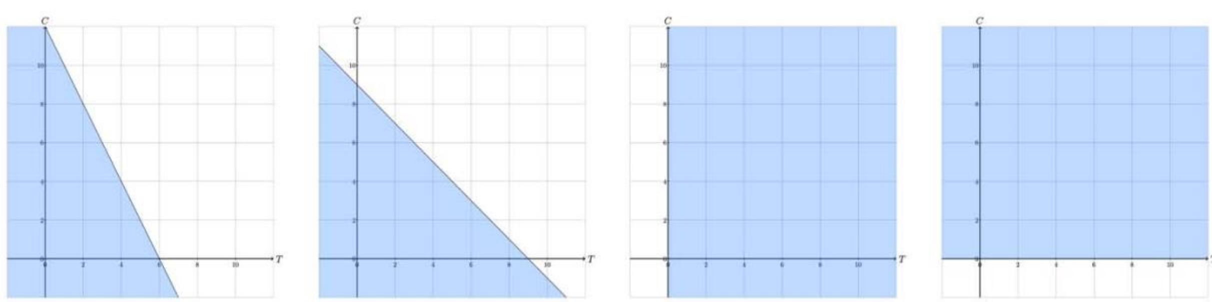


Figure 2: Half-planes described by the constraints of the Legos LP – left to right:
$2T+C\leq12$, $2T+2C\leq18$, $T\geq0$, $C\geq0$

as $2C+T \leq 12$, describes a half-plane: all the points on the line $2C+T=12$, along with the points to the left of (or below) this line.

Figure 1 illustrates the half-planes described by the four constraints of the Legos LP. For example, the shaded region in the leftmost graph in Figure 2 consists of all of the solutions to the Legos LP that satisfy constraint 1, or that do not require more than 12 large blocks.

The feasible region of a linear program is the set of solutions that satisfy all constraints. The feasible region of the Legos LP consists of the points in the intersection of all four half-planes described by its constraints, as illustrated in Figure 2. The feasible region of the Legos LP is illustrated in Figure 3. Thus, every point in the shaded region in Figure 3 represents a valid solution to the Legos problem.

A linear program is infeasible if its feasible region is empty (in other words, the linear program has no feasible solutions).

A feasible region is unbounded if one or more decision variables can increase (or decrease) indefinitely inside of the feasible region. In this case, the linear program may be unbounded, meaning that there is no optimal solution because the values of one or more variables can increase indefinitely.

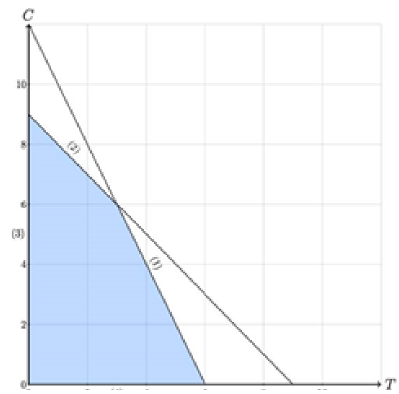
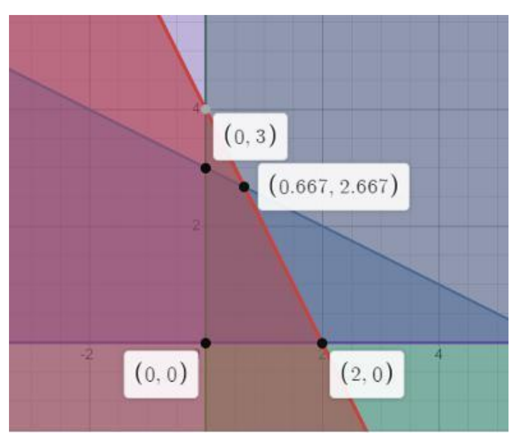


Figure 3: Feasible region of the Legos LP

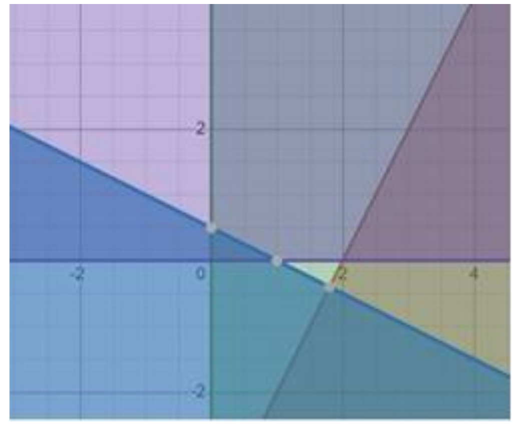
Problem: Suppose the inequalities are the constraints of a linear program. Graph the feasible region of each linear program. Is the associated linear program infeasible?

A. $2x+y \leq 4$ $x+2y \leq 6$ $x \geq 0$ $y \geq 0$
B. $-2x+y \leq -4$ $x+2y \leq 1$ $x \geq 0$ $y \geq 0$
C. $2x-y \leq -3$ $x-2y \leq 6$ $x \geq 0$ $y \geq 0$

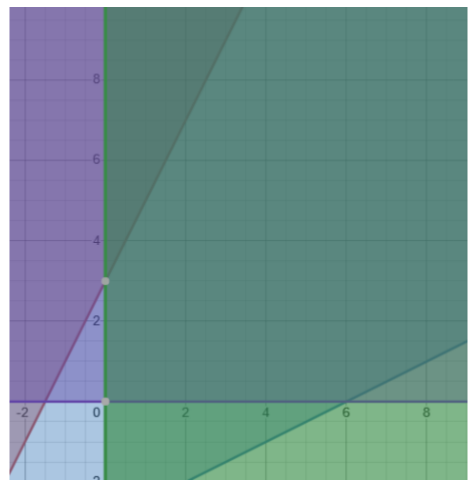
Solutions:
A. A graph of these constraints looks like the figure to the right. The feasible region is the polygon with vertices (0,0),(0,3), (0.667, 2.667), and (2,0). Any of the points in the region or on its boundary satisfy all four constraints.



B. This is an example of a system that is infeasible. The constraints do not create a region where all four solutions overlap. The system is illustrated in the figure to the right.



C. This is an example of an unbounded feasible region. The feasible region has corners at (0,0), (0,3), and (6,0), but extends without bound between the lines $2x-y=-3$ and $x=0$ as x and y increase to infinity.



Next, let's look at a similar problem, with a different setup: Rose is an engineering major and President of the Engineering Club at her college. The current club officers inherited 24 batteries (each of 1.5 V), 18 bulbs, and a bulk of electrical lead. A nearby high school needs some simple circuits and has funding to pay for the supplies. The high school principal reached out to the Engineering Club for help. The club officers agreed to help and at the same time wanted to generate funding for their club. The high school needs any of the following two different types of circuits:

- Type I Circuit with 2 battery cells (connected in series) and 1 bulb (Figure 3A)
- Type II Circuit with 2 battery cells (connected in series) and 2 bulbs connected in parallel (Figure 3B)

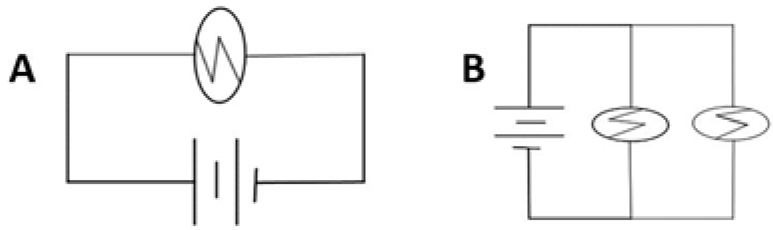


Figure 4 – Different circuits for the two circuits problem.

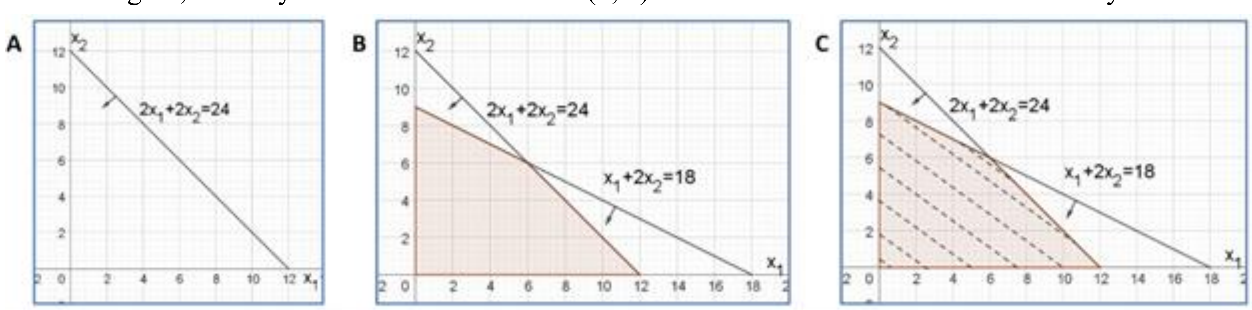
Rose and the club officers want to know how many circuits of each type they can make with the supplies they already have and make the maximum profit if they can sell the Type I circuit for \$8 and Type II circuit for \$11. Lily, Rose's friend, and mathematics major, who has taken an optimization class agrees to help them in finding a solution to their maximization problem. She explains to Rose that their problem is a *Linear Optimization Problem* with two unknowns:
C_1 - the number of Type I circuits the club officers should make,
C_2 - the number of Type II circuits the club officers should make.

and that the mathematical model consists of finding values of C_1 and C_2 that maximize the *linear objective function* (the profit function in this case)

$$f(C_1, C_2)=8C_1+11C_2$$

Subject to the constraints:
$2C_1+2C_2 \leq 24$ (The number of cells used in the circuits must be less than or equal to 24.)
$C_1 + 2C_2 \leq 18$ (The number of bulbs used in the circuits must be less than or equal to 18.)
$C_1 \geq 0$ (They can't make a negative number of circuits.)
$C_2 \geq 0$

Lily explains to the club officers how to solve the problem. First, she points that she will deal with the constraint of the batteries by drawing the line $2x_1+2x_2=24$ on a cartesian −coordinate plane and indicating by arrow the feasible half-space (the region that corresponds to all points (options) that are feasible with respect to the constraint of the batteries) (Figure 4A). Then she demonstrates to the club officers that the option of making 6 circuits of Type I and 3 of Type II is feasible hence point (6, 3) lies in the feasible half-space. Next, she repeats the same work for the constraint of the bulbs (by drawing the equation $x_1+2x_2=18$ and indicating by arrow its feasible half-space). Then she does the same for the last two (nonnegativity) constraints. Afterwards, she shades the feasible region (the intersection of the four half spaces) which is a polygon (Figure 4B) and explains to the officers that the shaded (feasible) region displays their options. Lily points out that the last step in solving their problem is plotting the objective function by picking any reasonable value and setting the profit function $f=8x_1+11x_2$ equal to that value. She plots the lines corresponding to several of these profit equations (called contour lines), which are all parallel (Figure 5C). Rose notices that the optimal solution for their problem is located at a vertex of the feasible region, and Lily confirms that the vertex (6, 6) is such solution that outlines how many circuits of



Type I (6) and Type II (6) the officers need to make to produce as much profit as possible ($114).

Figure 5: Geometric Solution of the Two Circuits Problem

The Legos model is a resource allocation linear program. Resource allocation problems typically seek to maximize profit or revenue given limited production resources such as materials, manpower, or time. Resource allocation was one of the first applications of linear programming and is still commonly used in industry

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Here are a few resource allocation problems for modeling practice. Hint: Before writing the models for these problems, ask yourself the following questions:

1. What are the decisions that need to be made in this problem? (Answering this question helps to define the decision variables.)

2. What is the goal in the problem? (Answering this question helps to define the objective function.)

3. What are the restricted resources? (This helps to define the constraints.)

Manufacturing Problem:

A manufacturing company can make devices using teams composed of 2 experienced engineers or 1 experienced engineer and three interns. Experienced teams build three devices every two hours, or 12 in an eight hour shift, while trainee teams build one device per hour, or eight in an eight hour shift. If the company has 12 engineers and 18 interns allocated to making devices, how many experienced teams and trainee teams should they use to maximize the number of devices constructed?

Solution: Let x be the number of engineer-only teams and let y be the number of training teams.
The model is:
Maximize $f(x,y)=12x + 8y$
Subject to constraints:
$2x+y\leq 12$
$3y\leq 18$
$x\geq 0$
$y\geq 0$

Another Manufacturing Problem:

A company makes mp3 players to sell. They make two types of players, an economy player that sells for $45 and a higher-end player that sells for $65. The players are assembled from a battery and circuit components. One team creates the circuits and a second team makes the batteries and assembles the players. The economy model requires 1.5 worker hours for each battery and 2 worker hours for each circuit assembly. The higher end model requires 2 worker hours for each battery and 3 worker hours for each circuit assembly. In the battery assembly area there are 560 worker hours available and in the circuit assembly area there are 780 worker hours available. How many of each type of system should be built to maximize revenue?

Solution: Let x be the number of economy players produced and y be the number of higher-end players.

Maximize f(x,y)=45x+65y
Subject to constraints:
$1.5x + 2y \leq 560$
$2x+3y \leq 780$
$x \geq 0$
$y \geq 0$

Three Circuits Problem: After two days, the principal of the high school calls Rose and asks that they might also consider making Type III Circuits with 2 battery cells (connected in series) and 3 bulbs connected in parallel (Figure 6)

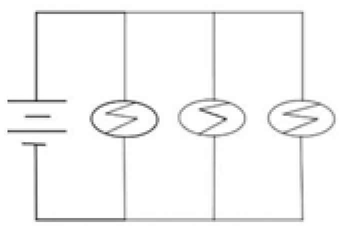

Figure 6-Circuit added for the Three circuit problem

Lily offers to help them again. The officers decide that they can charge $15 for Type III circuits, so Lily adds the variable x_3 (number of Type III circuits) to her mathematical model:

Maximize the objective profit function: $f(x_1, x_2, x_3) = 8x_1+11x_2+15x_3$

subject to the constraints:
$2x_1+2x_2+2x_3 \leq 24$
$x_1+2x_2+3x_3 \leq 18$
$x_1 \geq 0$
$x_2 \geq 0$
$x_3 \geq 0$

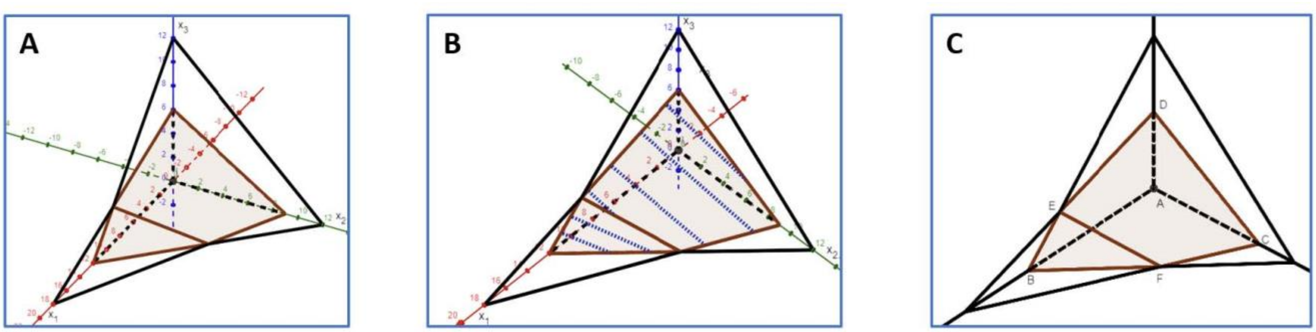

Figure 7 – 3 dimensional drawing of feasible region for three circuit problem.

The need for a new algorithm
The addition of a third variable has complicated the problem by making it three dimensional instead of two dimensional. This makes it more difficult to draw and solve. In high school algebra classes, you should have learned an algebraic method of solving this problem. In this module, we want to use the properties of the geographic representation of the problem to solve it.

Let’s look at the geometry more closely:
Constraint 1: $2T+C\leq 12$
Constraint 2: $2T+2C\leq 18$
Constraint 3: $T\geq 0$
Constraint 4: $C\geq 0$

In the case of the Lego’s linear program, there are six pairs of equations: (1) and (2), (1) and (3), (1) and (4), (2) and (3), and (2) and (4). Upon inspection, we see that all six pairs are linearly independent. For example, we can solve the system formed by equations (1) and (2):

$2T+C=12$ and $2T+2C=18$

to find the point of intersection of the two constraints: $(T,C)=(3,6)$. The intersection points obtained by solving all six of these systems of equations are as follows:

Constraints	Intersection Point
1 and 2	(3, 6)
1 and 3	(0, 12)
1 and 4	(6, 0)
2 and 3	(0, 9)
2 and 4	(9, 0)
3 and 4	(0, 0)

Let’s graph this feasible region of the Legos problem to see where all of these points reside in Figure 8.

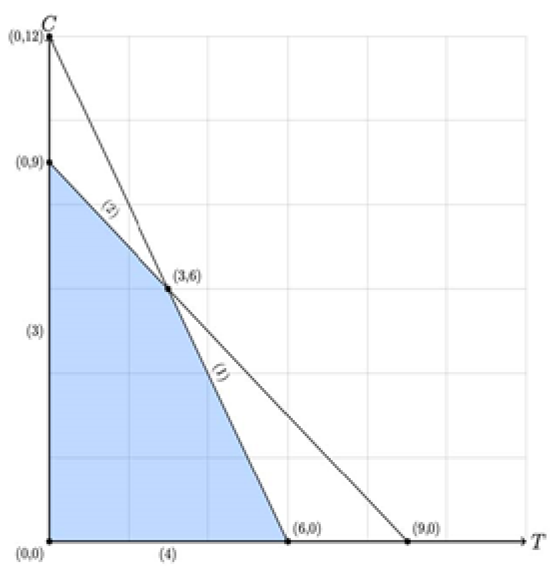

Figure 8

Notice that our process of solving every linearly independent pair of equations resulted in all of the corner points, or basic feasible solutions, of the Legos LP, but it also resulted in two other points: (0,12) and (9,0). We can see in Figure 8 that these two points are at the intersections of two constraint boundaries, but they are not in the feasible region. We can see both geometrically and algebraically that (9,0) violates constraint (1) and (0,12) violates constraint (2).

The points that are found at the intersections of constraints, such as the six that we found in the table above for the Legos problem, are called basic solutions.

The process described above leads us to the following theorem and algorithm for solving linear programs with bounded feasible regions.

Theorem 1. If P is a linear program with a bounded, nonempty, feasible region, then an optimal solution to P occurs at a basic feasible solution.

Corner Point Algorithm (to Solve an LP): Suppose linear program (P) has a bounded, nonempty feasible region. Further suppose that (P) has d decision variables. To solve (P):
i. Find all sets of d linearly independent constraints of (P)
ii. Solve each of the corresponding d x d linear systems, arriving at a set of basic solutions, S
iii. Test each basic solution for feasibility, eliminating the infeasible basic solutions. The remaining set of solutions, B, is the set of basic feasible solutions, or corner points, of the region.
iv. Evaluate the objective function P at each element of B. Apply Theorem 1 to identify an optimal solution to P.

Rationalizing the Simplex Algorithm: As we work on increasingly complex problems with more variables and more constraints, drawings will be of less use. Beyond 3D, direct visualization of data can be challenging and require specialized techniques. When we increase the number of variables, the number of points increases with the square of the number of variables. Although the algorithm we will soon introduce may seem computationally complex, it is easier to handle as the number of variables increases than solving for all of the basic solutions, determining which are feasible and plugging them all back into the objective function to determine which is the best solution.

If we look at Rose's three circuit problem, we can follow a geometric interpretation of the simplex algorithm and use that to explain what we are doing. Figure 7 shows the feasible region of the problem, with the vertices (basic feasible solutions) labelled A through F.

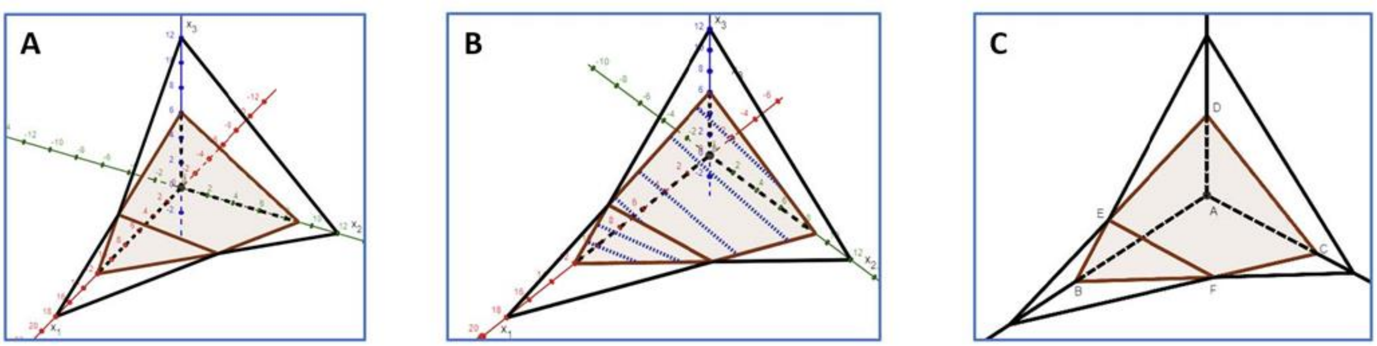

Figure 7 (again) – 3 dimensional drawing of feasible region for three circuit problem.

When solving a two-dimensional problem, the line and half-planes of each constraint overlapped to create a polygon feasible region. In three dimensions, plane and half-spaces overlap similarly to create a polyhedron feasible region. Feasible regions are convex and the optimal solution is found by following parallel contour lines (shown in figure 7B) from one feasible solution to a better solution until the optimal solution is found. In 1946 Georg Dantzig came up with the Simplex Algorithm to automate the finding of solutions to this problem. Geometrically, we can envision a "polytope" of the feasible region. That is, the vertices of the region with lines showing the edges between them. A polytope of Rose's three circuit problem is shown in figure 8.

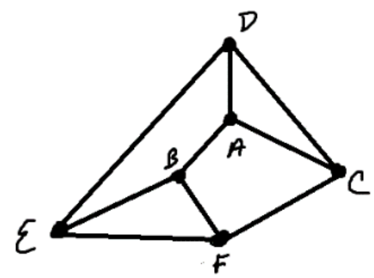

Figure 8 – A polytope of Rose's problem.

Note, it has a line from each vertex in the feasible region to each other vertex in the feasible region corresponding to an edge in Rose's graph. When looking at a polytope, each vertex is a **basic feasible solution.** If there is an optimal solution to the problem, it will be one of the vertices of this polytope. As we saw in the previous examples, narrowing the possible solutions to only the vertices of the feasible region makes the number of possible solutions finite, rather than infinite, but, as the size of the linear problem increases, the number of vertices may still become unmanageable to find and solve for all possible solutions. The Simplex Algorithm uses the fact that, if a basic feasible solution is not an optimal solution, there is at least one edge leading away from it to a better solution. By following these edges, we should find either the maximum value (our solution), or an unbounded edge, implying there is no solution.

Again, let's try to maximize the objective profit function:

$f(x_1, x_2, x_3) = 8x_1+11x_2+15x_3$

subject to the constraints:

$2x_1+2x_2+2x_3\leq 24$

$x_1+2x_2+3x_3\leq 18$

$x_1\geq 0$

$x_2\geq 0$

$x_3\geq 0$

We will introduce three slack variables, which represent the difference between the variable and the bound. In Rose's problem, the variables and their meanings are:

s_1 is the number of batteries that will remain after the club officers make some number of circuits.

s_2 is the number of bulbs that will remain after the club officers make some number of circuits,

z – the profit they will generate.

Express each slack variable in terms of the previous variables, x_1, x_2, and x_3.

$s_1=24-2x_1-2x_2-2x_3$ Call this A_1

$s_2=18-x_1-2x_2-3x_3$ Call this A_2

$z = 8x_1+11x_2+15x_3$ Call this A_z

Note the relationship between these equations and the constraints and profit function above.

Step 1) Lily starts with the vertex A (0, 0, 0) , and lets x1=0, x2=0, and x3=0.

Step 2) There are three edges incident with A, so she must determine which, AB, AC, or AD leads toward higher profit by examining the profit equation. All of the coefficients in this equation are positive, so increasing any of the variables will increase profit, but the highest coefficient is 15, in front of x_3, so increasing this variable will increase profit the most.

Step 3) Lily takes $x_1=0$ and $x_2=0$ and rewrites equations A_1, A_2, and A_Z

$s_1=24-2x_3$ Call this "new A_1"
$s_2=18-3x_3$ Call this "new A_2"
$z = 15x_3$ Call this "new A_z"

Step 4) These equations give us limits for x_3, since the slack variables can't be less than 0. That is new A_1 limits x_3 to be 12, and new A_2 limits x_3 to be 6. The conclusions should be that x_3 be increased to 6, which updates our solution to (0, 0, 6), with a profit z=90. In our polytype, this is vertex D.

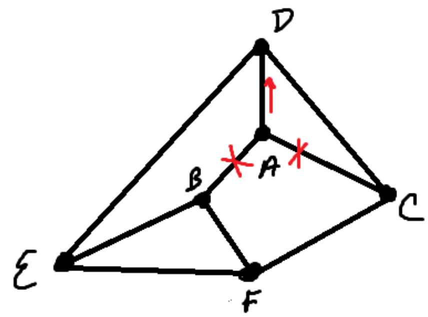

Figure 9-In terms of the polytope, Lily has moved from vertex A to vertex D as her solution.

Lily now has to repeat steps 1-4 applied to vertex D, to determine if it is the optimal solution, and if not, to determine which edge leads to a better solution.

She rearranges equations A_1, A_2, and A_z, such that x_1, x_2, and s_2 appear on the right

$x_3=6-13x_1-23x_2-13s_2$ Call this D_1
$s_1=12-43x_1-23x_2+23s_2$ Call this D_2
$z = 90+3x_1+x_2-5s_2$ Call this D_z

Looking at these equations, she sees that increasing x_1 from 0 to 9 will cause the greatest increase in the profit, and that will decrease x_3 from 6 to 3 and s_1 from 12 to 0. This gives her vertex E(9, 0, 3) as her new solution, with profit z = 117.

She now has to rearrange D_1, D_2, and D_z such that the variables x_2, s_1 and s_2 (the variables that correspond to the feasible polyhedron faces that intersect at vertex E) appear on the right.

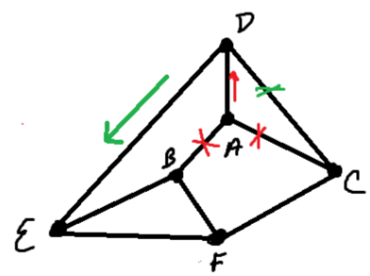

Figure 10-In this step, Lily moves to vertex E on the polytope and concludes that is her optimal solution.

$x_1=9-12x_2-34s_1+12s_2$ Call this E_1
$x_3=3-12x_2+14s_1-12s_2$ Call this E_2
$z = 117-12x_2-94s_1-72s_2$ Call this E_z

There are three edges incident with vertex E, but none of them will lead to an increase in profit (all coefficients in E_z are negative.) This leads her to conclude that (9, 0, 3) is the optimal solution, it has profit z = 117, $s_1=0$, and $s_2 = 0$.

Simplex Pivot Tools

Robert Vanderbei, Operations Research and Financial Engineering Professor at Princeton has created an online solver for the simplex algorithm. To see how this problem would be solved using this, or similar software, let's walk through the steps here.

Problems are set up to have three constraints and four variables, but you can change that in the header of the pivot tool. Once you have set the number of each, you can either choose to generate a random problem to work on, select an exercise from the book *Linear Programming: Foundations and Extensions* or enter the coefficients of the appropriate values into the *dictionary*-the table you will pivot to get the answer. Let's demonstrate that with an example problem.

A widget company manufactures two products, A and B. Product A sells for $10 of profit and product B sells for $12. The company has three departments which the products pass through. Department 1 requires 4 hours for each product A and product B. There are 600 hours of labor available in this department. Department 2 requires 3 hours for each product A and 2 hours for a product B. This department has 500 hours available. Department 3 requires 2 hours for each product A and 4 hours for product B. There are 500 hours available in this department

The objective function and constraints for the problem are:

$f(x_a, x_b) = 10\, x_a + 12\, x_b$

$4x_a + 4x_b \leq 600$
$3x_a + 2x_b \leq 500$
$2x_a + 4x_b \leq 500$

We introduce the slack variables w_1, w_2, and w_3 and rewrite the constraints as equations.

$4x_a + 4x_b + u = 600$
$3x_a + 2x_b + v = 500$
$2x_a + 4x_b + w = 500$

Solving for the slack variables, we have:

$w_1 = 600 - 4x_a - 4x_b$
$w_2 = 500 - 3x_a - 2x_b$
$w_3 = 500 - 2x_a - 4x_b$

Where,
$x_a, x_b, w_1, w_2, w_3 \geq 0$

The objective function and constraints are entered as the dictionary shown to the right. A dictionary is much like a tableau and is pivoted repeatedly until the solution can be read from the dictionary. In the dictionary, x_1 and x_2 correspond to x_a and x_b in the text. The variable x_2 has the greater of the two coefficients in the objective function, so we will pivot that with w_1 by clicking on the x_2 box in the w_1 row.

Current Dictionary

$$
\begin{aligned}
&\text{maximize} \quad \zeta = 10\,x_1 + 12\,x_2 \\
&\text{subject to:} \quad w_1 = 600 - 4\,x_1 - 4\,x_2 \\
&\qquad\qquad\quad w_2 = 500 - 3\,x_1 - 2\,x_2 \\
&\qquad\qquad\quad w_3 = 500 - 2\,x_1 - 4\,x_2 \\
&\qquad x_1 \;\; x_2 \;\; w_1 \;\; w_2 \;\; w_3 \geq 0
\end{aligned}
$$

Figure 11-The starting dictionary for the widget manufacturer

This changes our dictionary to look like the one on the right. We still need to pivot x_1, and the tool recommends pivoting it with w_3 as shown by the highlight in the w_3 row. Clicking on the x_1 box in the line starting with w_3= gives us the final dictionary to the right. From this dictionary, we can read the solution to our problem, $x_2 = 100$, $x_1 = 50$ and our total profit is 1700.

Current Dictionary

$$
\begin{aligned}
&\text{maximize} \quad \zeta = 1800 + -2\,x_1 + -3\,w_1 \\
&\text{subject to:} \quad x_2 = 150 - 1\,x_1 - 1/4\,w_1 \\
&\qquad\qquad\quad w_2 = 200 - 1\,x_1 - -1/2\,w_1 \\
&\qquad\qquad\quad w_3 = -100 - -2\,x_1 - -1\,w_1 \\
&\qquad x_1 \;\; x_2 \;\; w_1 \;\; w_2 \;\; w_3 \geq 0
\end{aligned}
$$

Figure 12-The ending dictionary for the widget manufacturer

Pivot Table Problems:

1. Use the pivot table to solve the Lego problem which we initially had: Imagine that you are running a manufacturing business that makes tables and chairs out of Legos. The tables can be sold for $16 each and the chairs are sold for $10 each. Each table requires 2 big blocks and 2 little b locks while each chair requires 1 big block and 2 little blocks, as shown below. There are 12 big and 18 small blocks available to build tables and chairs.

Solution: Starting and final pivot tables should look like:

Current Dictionary

$$
\begin{aligned}
&\text{maximize} \quad \zeta = 16\,x_1 + 10\,x_2 \\
&\text{subject to:} \quad w_1 = 12 - 2\,x_1 - 1\,x_2 \\
&\qquad\qquad\quad w_2 = 18 - 2\,x_1 - 2\,x_2 \\
&\qquad x_1 \;\; x_2 \;\; w_1 \;\; w_2 \geq 0
\end{aligned}
$$

Current Dictionary

$$
\begin{aligned}
&\text{maximize} \quad \zeta = 108 + -6\,w_1 + -2\,w_2 \\
&\text{subject to:} \quad x_1 = 3 - 1\,w_1 - -1/2\,w_2 \\
&\qquad\qquad\quad x_2 = 6 - -1\,w_1 - 1\,w_2 \\
&\qquad x_1 \;\; x_2 \;\; w_1 \;\; w_2 \geq 0
\end{aligned}
$$

Giving the solution, make 3 tables and 6 chairs to sell for $108.

2. Use the pivot table to solve the two circuit problem from earlier. The club can make either

- Type I Circuit with 2 battery cells (connected in series) and 1 bulb (Figure 3A)
- Type II Circuit with 2 battery cells (connected in series) and 2 bulbs connected in parallel (Figure 3B)

Rose and the club officers want to know how many circuits of each type they can make with the supplies they already have and make the maximum profit if they can sell the Type I circuit for $8 and Type II circuit for $11. The club has 24 batteries (each of 1.5 V), 18 bulbs, and a bulk of electrical leads

.

Solution: Starting and final pivot tables should look like:

Current Dictionary

maximize $\zeta = 8\,x_1 + 11\,x_2$

subject to:

$w_1 = 24 - 2\,x_1 - 2\,x_2$

$w_2 = 18 - 1\,x_1 - 2\,x_2$

$x_1 \;\; x_2 \;\; w_1 \;\; w_2 \geq 0$

Current Dictionary

maximize $\zeta = 114 + -3\,w_2 + -5/2\,w_1$

subject to:

$x_2 = 6 - 1\,w_2 - -1/2\,w_1$

$x_1 = 6 - -1\,w_2 - 1\,w_1$

$x_1 \;\; x_2 \;\; w_1 \;\; w_2 \geq 0$

Giving the solution, make 6 of each circuit for a total price of $114.

3. A manufacturing company can make devices using teams composed of 2 experienced engineers or 1 experienced engineer and three interns. Experienced teams build three devices every two hours, or 12 in an eight hour shift, while trainee teams build one device per hour, or eight in an eight hour shift. If the company has 12 engineers and 18 interns allocated to making devices, how many experienced teams and trainee teams should they use to maximize the number of devices constructed?